Name: Solutions
Start Time:
End Time:
Date:

Math 260
Quiz 7 (30 min)

1. (2 points) Use Cramer's rule to solve the system of equations below:

$$\begin{aligned} 5x_1 + x_2 - x_3 &= -7 \\ 2x_1 - x_2 - 2x_3 &= 6 \\ 3x_1 \qquad + 2x_3 &= -7 \end{aligned}$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} -7 & 1 & -1 \\ 6 & -1 & -2 \\ -7 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{vmatrix}} = \frac{23}{-23} = -1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 5 & -7 & -1 \\ 2 & 6 & -2 \\ 3 & -7 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{vmatrix}} = \frac{92}{-23} = -4$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 5 & 1 & -7 \\ 2 & -1 & 6 \\ 3 & 0 & -7 \end{vmatrix}}{\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -2 \\ 3 & 0 & 2 \end{vmatrix}} = \frac{46}{-23} = -2$$

$$\boxed{\text{So } x_1 = -1,\ x_2 = -4,\ x_3 = -2}$$

2. (2, 1, 3, 1, 1 points) Let $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

a) Find the characteristic polynomial of A
b) Find the eigenvalues of A including multiplicities
c) Find all eigenvectors of A
d) Find the diagonalizing matrix P
e) Find the diagonal matrix D where $D = P^{-1}AP$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} \lambda-3 & -1 & -1 \\ 4 & \lambda+2 & 5 \\ -2 & -2 & \lambda-5 \end{bmatrix}$$

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a) $c_A(x) = |xI - A| = \begin{vmatrix} x-3 & -1 & -1 \\ 4 & x+2 & 5 \\ -2 & -2 & x-5 \end{vmatrix}$

Expand along 1st row $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$= +(x-3)\begin{vmatrix} x+2 & 5 \\ -2 & x-5 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 5 \\ -2 & x-5 \end{vmatrix} + (-1)\begin{vmatrix} 4 & x+2 \\ -2 & -2 \end{vmatrix}$$

$$= (x-3)\left[(x+2)(x-5) - (-2)(5)\right] + 1\left[4(x-5) - (-2)(5)\right] - 1\left[4(-2) - (-2)(x+2)\right]$$

$$= (x-3)(x+2)(x-5) + 10x - 30 + 4x - 20 + 10 + 8 - 2x - 4$$

$$= (x-3)(x+2)(x-5) + 12x - 36 = (x-3)(x+2)(x-5) + 12(x-3)$$

$$= (x-3)\left[(x+2)(x-5) + 12\right] = (x-3)\left[x^2 - 3x - 10 + 12\right] = (x-3)(x^2 - 3x + 2)$$

$$= (x-3)(x-2)(x-1)$$

$\boxed{\text{So} \quad c_A(x) = (x-3)(x-2)(x-1)}$

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b) $c_A(x) = 0$

$(x-3)(x-2)(x-1) = 0$

$x = 3, \; x = 2, \; x = 1$

Eigenvalues	Multiplicities
$\lambda = 3$	1
$\lambda = 2$	1
$\lambda = 1$	1

c) Eigenvectors for $\lambda = 3$

$$(\lambda I - A)\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 4 & 5 & 5 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$x_1 = 0$

$x_2 + x_3 = 0 \implies x_2 = -t$

$$\boxed{\text{Eigenvectors for } \lambda = 3 = \left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \,\middle|\, \begin{array}{l} t \in \mathbb{R} \\ t \neq 0 \end{array} \right\}}$$

Since $\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, a basic set of eigenvectors for $\lambda = 3$ is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Eigenvectors for $\lambda = 2$

$$(\lambda I - A)\vec{x} = \vec{0} \implies \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 4 & 4 & 5 & 0 \\ -2 & -2 & -3 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + x_2 = 0 \implies x_1 = -t$

$x_3 = 0$

$$\boxed{\text{Eigenvectors for } \lambda = 2 = \left\{ \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} \,\middle|\, \begin{array}{l} t \in \mathbb{R} \\ t \neq 0 \end{array} \right\}}$$

Since $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, a basic set of eigenvectors for $\lambda = 2$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Eigenvectors for $\lambda=1$

$(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 4 & 3 & 5 & 0 \\ -2 & -2 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = t$

$x_1 - x_3 = 0 \Rightarrow x_1 = t$

$x_2 + 3x_3 = 0 \Rightarrow x_2 = -3t$

$$\boxed{\text{Eigenvectors for } \lambda=1 = \left\{ \begin{bmatrix} t \\ -3t \\ t \end{bmatrix} \middle| \begin{array}{l} t \in \mathbb{R} \\ t \neq 0 \end{array} \right\}}$$

Since $\begin{bmatrix} t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$, a basic set of eigenvectors for $\lambda=1$ is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$

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d) $$\boxed{P = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}}$$

e) $$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$